8.1. Actions of discrete groups. Let Γ be a group acting on a topological space X by homeomorphisms.

(a) Show that Γ acts properly discontinuously if and only if the map $\Gamma \times X \to X \times X$ given by $(\gamma, x) \mapsto (\gamma x, x)$ is *proper*, that is, the inverse image of any compact set is compact. Here Γ is equipped with the discrete topology.

(b) Show that if X is a topological manifold and Γ acts freely and properly discontinuously, then the quotient space X/Γ is a topological manifold and the projection $\pi\colon X \to X/\Gamma$ is a covering map. (We assume topological manifolds to be Hausdorff.)

(c) Suppose now that X, M are connected topological manifolds, $F\colon X \to M$ is a covering map, and $\Gamma \subset \mathrm{Homeo}(X)$ is the group of deck transformations of F. Show that Γ acts freely and properly discontinuously on X.

8.2. Translations. Suppose that Γ is a group of translations of $\mathbb{R}^m$ that acts freely and properly discontinuously on $\mathbb{R}^m$.

(a) Show that there exist linearly independent vectors $v_1, \dots, v_k \in \mathbb{R}^m$ such that

$$\Gamma = \left\{ x \mapsto x + \sum_{i=1}^{k} z_i v_i : (z_1, \dots, z_k) \in \mathbb{Z}^k \right\} \simeq \mathbb{Z}^k.$$

(b) Let l denote the infimum of the lengths of all closed curves in $\mathbb{R}^m/\Gamma$ that are not null-homotopic. Show that l equals the length of the shortest non-zero vector of the form $\sum_{i=1}^{k} z_i v_i$ with $z_i \in \mathbb{Z}$ as above.

8.3. Some consequences of non-positive sectional curvature. Let M be a Hadamard manifold. Prove the following:

(a) For each $p \in M$, the map $(\exp_p)^{-1}\colon M \to TM_p$ is 1-Lipschitz.

(b) For $p, x, y \in M$, it holds

$$d(p,x)^2 + d(p,y)^2 - 2d(p,x)d(p,y)\cos\gamma \le d(x,y)^2,$$

where γ denotes the angle in p.

(c) Let M be a complete manifold of non-positive sectional curvature with universal covering $\pi\colon \tilde{M} \to M$. If there is a geodesic $\tilde{\gamma}\colon \mathbb{R} \to \tilde{M}$ which is invariant under deck transformations, then M is not compact.